Algebra and Number Theory
Individual Oral Test

1. Consider $f \in \mathbb{Z}_{>0}$ and nonzero vector spaces V_i indexed by $i \in \mathbb{Z}/f\mathbb{Z}$. Suppose that there are linear maps $\phi_i : V_i \to V_{i+1}$ and $\psi_i : V_i \to V_{i-1}$ such that

$$\phi_{i-1} \circ \psi_i = 0, \quad \psi_{i+1} \circ \phi_i = 0.$$

(We may think of a circular graph with oriented edges such that the "Orpheus condition" holds: *Whenever you turn back while traveling through the graph you are killed.*)

Prove that there exists lines $\ell_i \subset V_i$ for every $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$\phi_i(\ell_i) \subset \ell_{i+1}, \quad \psi_i(\ell_i) \subset \ell_{i-1}$$

under one of the following two conditions:

1. all $\psi_i = 0$, or
2. $\dim V_i$ are equal to each other.

Hint: use induction

2. For k non-negative integer, let $V_k := \mathbb{R}[x]_{\leq k}$ be the vector space of real polynomials of degree at most k with an action by $\mathrm{SL}_2(\mathbb{R})$ by

$$\gamma \cdot P(x) = (cx+d)^k P\left(\frac{ax+b}{cx+d}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

1. Show that V_k is an irreducible representation of $\mathrm{SL}_2(\mathbb{R})$;

2. For non-negative integers m, n, consider $V_{m,n} := V_m \otimes V_n$ as a subspace of $\mathbb{C}[x,y]$ of polynomials with both x, y-degrees at most k. Assume $m \geq n \geq 1$. Show that following exact sequence is exact and split as representations of $\mathrm{SL}_2(\mathbb{R})$.

$$0 \longrightarrow V_{m-1,n-1} \xrightarrow{\cdot(y-x)} V_{m,n} \xrightarrow{y=x} V_{m+n} \longrightarrow 0.$$

This implies the following decomposition of representations:

$$V_m \otimes V_n = \oplus_{i=0}^{n} V_{m+n-2i}.$$

3. For non-negative integers $\ell \geq m \geq n$ consider the space of invariants $(V_\ell \otimes V_m \otimes V_n)^{\mathrm{SL}_2(\mathbb{R})}$. Show that this space is either trivial or one-dimensional; it is non-trivial if and only if

$$\ell + m + n \equiv 0 \mod 2, \qquad \ell + m \geq n.$$

3. Is $(x^2+3)(x^3+2)$ solvable mod p for every prime p?